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MOMENT DESCRIPTION OF GAS MIXTURES-I

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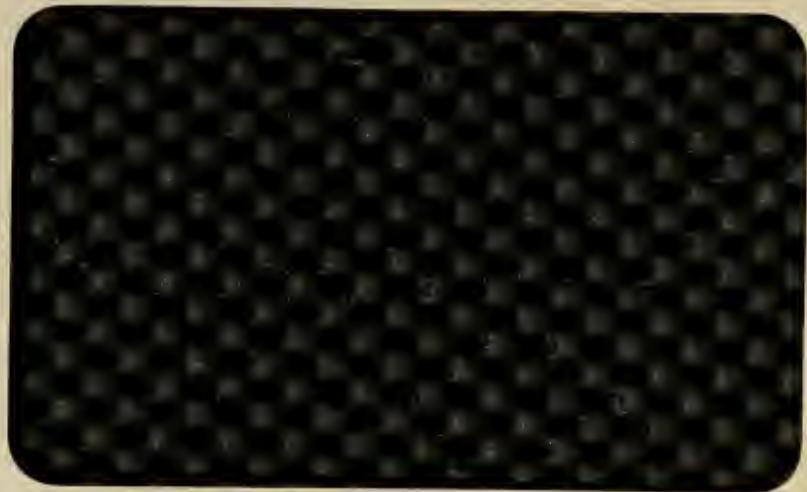The University of New Mexico
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ABSTRACT

A 13n moment description of n'ary gas mixtures can be con-
structed for the Boltzmann Equation using a procedure due to
Grad. The densities, flow velocities, specific energies, stress
tensors and heat flow vectors of individual components are con-
sidered to be fundamental quantities in this description. To
achieve a determined set of macroscopic equations, the component
velocity distribution function is approximated by means of a
product of an exponential and a third degree polynomial which
reduces to a Maxwellian distribution when stresses and heat
flows are neglected, and otherwise yields the same fundamental
13 moments as the exact distribution function.

The resulting moment equation includes various gas-mixture
theories as special cases. On neglecting the spacial dependence
(i.e., for local study) one obtains the theory of relaxation
times. If stresses and heat flows are neglected and the temp-
eratures are set equal, one obtains the Euler equations and their
generalizations. Lastly, if stress, heat flows, temperature
differences, and component velocity differences are assumed to
be of first order, then it follows that a) the temperature
differences are, in case of uncharged fluids, of second order
and thus can be neglected; b) the stresses, heat flows and

diffusional velocity can be expressed in terms of the thermo-
dynamic variables, leading to the Chapman-Enskog theory of
material constants and to the reduction of the system to Navier-
Stokes equations complemented by a diffusion relation. For a
plasma, the temperature differences are of first order, and thus
one obtains a description involving individual temperatures of
ions and electrons.

Collisional terms in the moment equations were evaluated
exactly in terms of certain integrals depending on the type of
interaction. The latter were evaluated explicitly for the case
of inverse power interaction laws (including elastic sphere
model). In the case of charged components, the Coulomb inter-
action with a cut-off at Debye radius was assumed.

For plasma, a marked simplification in the collisional terms
is achieved by assuming that the ratio of electron mass to ion
mass is small and by dropping terms multiplied by this ratio.

TABLE OF CONTENTS

Introduction

The object of this note is to supply certain results derived from kinetic theory concerning the macroscopic description of mixture of gases. The description proposed is special in that it is entirely based on the microscopic description of the mixture via Boltzmann's Equation coupled to the Maxwell's Equations. Yet it is sufficiently broad to include various theories at present in use as limiting cases. Some of these will be discussed in subsequent notes.

The contents are based to a large extent on authors dissertation [7] to which the reader is referred for derivations. (In one instance the derivation is little known and not readily available, and is therefore reproduced in the Appendix.) The background for this work is most readily found in [1], [2], [3].

In order to enhance readability we have adopted a notation which is not consistent throughout the paper. Index notation has been avoided but not altogether eliminated. When from the meaning of a symbol its tensorial character is understood, all reference to this character has been omitted. Dots between symbols indicate contractions. Thus we write $u \cdot \frac{\partial}{\partial x}$ for $u^j \frac{\partial}{\partial x_j}$, $\frac{\partial}{\partial x} \cdot u$ for $\frac{\partial}{\partial x_j} u^j$, while $\frac{\partial}{\partial x} u$ denotes the tensor $\frac{\partial}{\partial x_i} u^j$. When a tensorial index appears under the differentiation bar it is meant as a rule to be the first index. Thus in equation (2.1) the expression $v \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial \phi}{\partial v}$ stands for $v^i \frac{\partial u^j}{\partial x_i} \frac{\partial \phi}{\partial v^j}$. In equation (4.3)

the terms $Q \frac{\partial}{\partial x} \cdot u$, $Q \cdot \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial x} \cdot Q$ represent, according to the above rule, $Q^i \frac{\partial u^j}{\partial x_j}$, $Q^j \frac{\partial u^i}{\partial x_j}$ and $\frac{\partial u^j}{\partial x_i} Q^j$ respectively. A cross product between a vector and a second rank tensor is to be read as follows:

$$(P \times B) = e_{jk\ell} \, P^{ik} \, B^\ell ,$$

$$(B \times P) = e_{ik\ell} \, B^k {}_P{}^{\ell j} .$$

Symbols for macroscopic quantities are, with noted exceptions, the same as those used in [5]. On occasion the same symbol is used for two different entities.

Equations in the text are numbered consecutively within Sections. The definitions are numbered consecutively with letter D preceding the number.

List of Symbols

(Parenthetical notes refer to definitions in the text)

1. Symbols occurring in the Boltzmann, Maxwell, and transfer equations.

$$f_r \quad (A)$$

$$\xi, \; \xi_1, \; \xi', \; \xi_1', \; v, \; c_r \; c, \; c^{(k)}, \; w$$

$$E \quad (1.1)$$

$$\mathcal{B}, \; B \quad (1.1)$$

$$F, \; F^i \quad (1.2)$$

$$\emptyset, \; \overline{\emptyset} \quad (B)$$

$$J_{rs}, \; J_{rs}(\emptyset), \; J_{rs}^{(k)}, \; \overline{J}_{rs}^{(k)} \quad (1.3)$$

$$V_{rs}(|w|, \theta) \quad (\text{Sec. 5})$$

$$\alpha, \; \theta, \; \chi, \; \epsilon, \; b, \; \beta \quad (\text{Sec. 5})$$

2. Moments, combinations of moments.

$$f_r^{i_1 i_2 \cdots i_k}, \; f_r^{(k)} \quad (\text{Sec. 3})$$

$$F_r^{(k)}, \; \overline{F}_r^{(k)}, \; F^{(k)} \quad (\text{Sec. 3})$$

$$\nu_r, \; \nu \quad (A) \; (D1)$$

$$\rho_r, \; \rho \quad (D1)$$

$$g_r, \; g \quad (D1)$$

$$u_r, \; u, \; (D1), \; u_r = u_r - u, \; v_{12} = v_1 - v_2$$

$$J_r, \; J, \; J_u \quad (D1)$$

$$e_r, \; \overline{e}_r, \; e \quad (\text{Sec. 3})$$

$$p_r, \; \overline{p}_r, \; p \quad (\text{Sec. 3})$$

$$P_r, \; \overline{P}_r, \; P \quad (\text{Sec. 3})$$

$$Q_r, \; \widetilde{Q}_r, \; \overline{Q}_r, \; Q \quad (\text{Sec. 3})$$

$$T_r, \; T, \quad (\text{Sec. 3})$$

σ, τ (D11)

F_r (D6)

3. Material constants

$m_r, m_{rs}, m_{10}, m_{20}$ (D1),(D10),(d11)

ε_r, Z (D1)

γ_r (D1)

U_{rs} (Sec. 5)

d

d_r, d_{rs} (Sec. 5.2)

χ_{rs} (Sec. 5.1)

μ, η (Sec. 5.1)

$\mathcal{A}_1, \mathcal{A}_2$ (Sec. 5.1)

$A_{rs}, A_{rs}^k, a_{rs}^k$ (Sec. 5) (Sec. 5.3)

$B_{rs}, B_{rs}^k, b_{rs}^k$ (Sec. 5) (Sec. 5.3)

d_o (Sec. 5.3)

z (Sec. 5.3)

χ (8.4)

4. Physical constants

ε = electron charge

m = electron mass

M = ion mass

k = Boltzmann's constant

c = velocity of light

γ = .5772... - Euler's constant

5. Special Symbols

$D, \bar{D}$ (D6) (D8)

$I = \delta_{ij}$ (the unit tensor)

e_{ijk}

$\{ \ \}$ (D7)

-, + (as subscripts).

1. Boltzmann Equations for a Gas Mixture

We consider a gas composed of n homogeneous components. Each component is described microscopically by its <u>number density function</u> f_r, r=1,2,...,n,[1] a function of position, velocity and time (x, ξ, t). (In all our considerations the arguments x and t will remain unchanged; to show the value of ξ at which f_r is evaluated, we shall write $f_r(\xi)$ for f_r.) The meaning of f_r is that

(A) $$\int f_r d\xi = \nu_r = \text{number density of component r},$$

(B) $$\int \emptyset f_r d\xi = \nu_r \overline{\emptyset},$$

where $\emptyset = \emptyset(x, \xi, t)$ is a "molecular property" function[2], while $\overline{\emptyset} = \overline{\emptyset}(x, t)$ is its average value (a state variable).

We use the following notation for various intrinsic properties (such as molecular mass, charge, etc.) and state variables:

---

[1]Some workers, see e.g. [2], prefer to consider the <u>mass density function</u> $m_r f_r$ instead of the number density. While this is a matter of taste, the use of the former will simplify the notation in the study of a single gas, while the latter will lead to simplifications in the case of mixtures.

[2]The word molecule is used here for particles of any kind.

$$(D1) \begin{cases} m_r = \text{mass of particle} \\ \epsilon_r = \text{charge} \\ \gamma_r = \epsilon_r/m_r \\ \rho_r = m_r \nu_r = \text{component (mass)}^3 \text{ density} \\ q_r = \epsilon_r \nu_r = \text{component charge density} \\ u_r = \overline{\xi} = \text{component velocity} \\ J_r = q_r u_r = \text{component current (density)} \\ \\ \nu = \sum_r \nu_r = \text{number density} \\ \rho = \sum_r \rho_r = \text{(mass) density} \\ q = \sum_r q_r = \text{charge density} \\ u = \frac{1}{\rho} \sum_r \rho_r u_r = \text{mass velocity (of the mixture)} \\ J = \sum_r J_r = \text{current (density)} \end{cases}$$

The charges and currents will produce an electromagnetic field $(E, \beta = cB)$ governed by Maxwell's Equations

$$(1.1) \begin{cases} \nabla \cdot E = 4\pi q \\ \nabla \times E + \dfrac{\partial B}{\partial t} = 0 \\ \nabla \cdot B = 0 \\ c^2 \nabla \times B - \dfrac{\partial E}{\partial t} = 4\pi J \end{cases}$$

Here c denotes the velocity of light.

The densities f_r satisfy the Boltzmann Equations

$$(1.2) \qquad \left(\frac{\partial}{\partial t} + \xi \cdot \frac{\partial}{\partial x} + (F + \gamma_r E + \gamma_r \xi \times B) \cdot \frac{\partial}{\partial \xi} \right) f_r = \sum_s J_{rs} .$$

---

[3] Words in parentheses are usually omitted.

Here, F is the outside (or body) force per unit volume (such as the force of gravity) and J_{rs} is the collision term expressing the rate at which the number of molecules of type (r,x,ξ) increases due to binary "collisions" with molecules of type s.[4] One shows that

$$(1.3) \qquad J_{rs} = \int d\xi_1 \, d\Omega \, \left(f_r(\xi') \, f_s(\xi_1') - f_r(\xi) f_s(\xi_1) \right)$$

where (ξ', ξ_1') are asymptotic velocities after an encounter of a molecule (r,ξ) with a molecule (s,ξ_1) with impact parameter b, and $d\Omega$ is the differential collisional cross-section for such an encounter, while the $d\Omega$ integration ranges over the impact parameter space. Explicit expressions for $d\Omega$, ξ', ξ_1' are considered

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[4] The controversy concerning the appropriateness of Boltzmann Equations for the determination of the f_r is beyond the scope of this paper. The terms on the left hand side are usually called drift terms. The electromagnetic term $\gamma_r(E + \xi \times B) \cdot \frac{\partial}{\partial \xi} f_r$ actually is not a drift term since it represents the change in f_r due to encounters with distant molecules and should be lumped with the collisional terms representing changes due to short range encounters. In case of short range intermolecular forces such as occur in encounters between neutral particles one usually avoids splitting into short and long range effects. In the case of interactions governed by Coulomb's law this splitting is necessary since then the encounters between distant molecules cannot be considered binary. The splitting is rather crude and consists in distinguishing a total effect on molecules by far away (beyond some shielding distance) molecules represented by the electromagnetic force that they produce, from effect by close (within the shielding distance) molecules considered to be involved in binary encounters.

in Section 6.

Together with the Maxwell's Equations, the Boltzmann Equations form a determined set of equations for the f_r, E and B. It is assumed that the properly posed problem for this system is the initial value problem: given f_r at $t = 0$, to determine f_r for $t > 0$.[5]

2. Equations of transfer. Equations of fluid dynamics

On multiplying the Boltzmann Equations by a molecular property $\phi(x, \xi, t)$ and integrating with respect to ξ, we obtain the equation of transfer of property ϕ. We consider only the case when $\phi = \phi(v)$, $v = \xi - U(x, t)$. Three cases of special interest are thus included, namely $U = 0$, $U = u_r$, and $U = u$. In these cases we shall use the symbols ξ, c_r, and c respectively in place of v. We call c_r the specific molecular component velocity and c the specific molecular velocity.

After some manipulations one gets the component transfer equation,

$$(2.1) \qquad (\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \cdot U) \nu_r \overline{\phi} + \frac{\partial}{\partial x} \cdot \nu_r \overline{v\phi} + \nu_r \overline{(v \cdot \frac{\partial U}{\partial x} - \gamma_r(v \times B)) \cdot \frac{\partial \phi}{\partial v}}$$

$$+ \nu_r \left[(\frac{\partial}{\partial t} + U \cdot \frac{\partial}{\partial x}) U - F - \gamma_r(E + U \times B) \right] \cdot \overline{\frac{\partial \phi}{\partial v}} = \sum_s J_{rs}(\phi),$$

where

$$(D2) \qquad J_{rs}(\phi) = \int \phi J_{rs} d\xi.$$

The total transfer equation is obtained by summing the above over r.

[5] The existence of solution proof has not yet been established in any generality.

The collisional terms do not vanish in general. One shows however that

$$\sum_s J_{rs}(m_r) = 0 \qquad r = 1,2,\ldots,n$$

$$\sum_{rs} J_{rs}(m_r \xi) = 0$$

$$\sum_{rs} J_{rs}(m_r \xi^2) = 0 \quad ^6$$

and that these are the only independent vanishing combinations. Correspondingly, the component transfer equation with $\emptyset = m_r$, and the total transfer equations with $\emptyset = m_r c$ and $\emptyset = \frac{1}{2} m_r c^2$ do not contain collisional terms. These are, explicitly,

$$(2.2) \left\{ \begin{array}{ll} (C_r) & (\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \cdot u_r)\rho_r = 0 \qquad r = 1,2,\ldots,n \\[2mm] (M) & \rho(\frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial x})u + \frac{\partial}{\partial x} \cdot P = \rho F + qE + J \times B \\[2mm] (E) & (\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \cdot u)e + \frac{\partial}{\partial x} \cdot Q + P : \frac{\partial u}{\partial x} = J_u \cdot (E + u \times B) \end{array} \right.$$

Here we used the following abbreviations:

$$(D3) \left\{ \begin{array}{l} \sum_r m_r \int c c f_r d\xi = P = \text{ flux of momentum observed while} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{moving with the fluid velocity u} \\[2mm] \sum_r \frac{1}{2} m_r \int c^2 f_r d\xi = e = \text{ intrinsic kinetic energy of the} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{mixture} \\[2mm] \sum_r \frac{1}{2} m_r \int c^2 c f_r d\xi = Q = \text{ flux of intrinsic kinetic energy} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{observed when moving with the fluid} \\[2mm] J_u = J - qu = \sum_r q_r(u_r - u) = \text{ conduction current} \end{array} \right.$$

[6] These relations express the conservation of mass, momentum and kinetic energy in a collision. $J_{rs}(m_r)$, $J_{rr}(m_r \xi)$, $J_{rr}(m_r \xi^2)$ vanish individually.

Integration of these equations with the conservation equations of fluid dynamics compels us to identify P, e, and q with the stress tensor, the internal energy and heat flow vector respectively[*].

We also have a relation between the components of P and e, namely

$$\tfrac{1}{3} P_{ii} = \tfrac{1}{3} P^{ii} = p = \text{scalar pressure.}$$

Defining the kinetic temperature T by

$$\tfrac{3}{2} kT = \tfrac{e}{\rho} , \qquad\qquad k = \text{Boltzmann constant}$$

the above relation becomes

$$p = \rho kT.$$

P and q can be, in principle, obtained from Maxwell's Equations (1.1) in terms of the ρ_a and the c_a. Substituting these in the fluid equations (1.2) we now get a set of 5×11 equations for the 5×4 unknowns ρ_a, u_a, P and q. P, being a symmetric tensor, has only 6 independent components; e is not a new unknown since it is expressed in terms of the trace of P. The system (1.1)–(2.2) is thus underdetermined and one of the principal objectives of the Kinetic Theory is to close the system by adjoining to it additional equations. In (1.2) the 5×11 variables ρ_a, u and e are differentiated with respect to time, while the others are not. These — the so-called thermodynamic variables — are also the only relevant ones

*This puts some light on the type of mixture described by the Boltzmann Equations. The gas is such that the intermolecular interactions do not contribute directly to the stress tensor, internal energy and the heat flow vector.

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in equilibrium[8]. It is the result (if not necessarily the intent) of the theory of normal solutions of Boltzmann Equations that the remaining $(3n+5)$ variables can be expressed (approximately) in terms of the primary $(n+4)$ variables, thus closing the system. Unfortunately the normal solutions give an accurate result only when the relaxation times are short, in particular when the equipartition of intrinsic kinetic energy per molecule is rapidly achieved. Indeed there is no provision in these solutions for non-equipartition; component temperatures turn out to be equal at all times.

On the other hand one knows of important phenomena such as those occurring in plasma at high temperature in which the equipartition of intrinsic kinetic energy is not likely to be achieved. To obtain a reasonable description of such phenomena we must then give up the use of normal solutions and build a finer theory.

3. Moments. Moment Equations

The problem of finding the density function f_r may be replaced by the equivalent problem of finding all its moments, i.e. the macroscopic tensors $f_r^{(k)}$ defined by

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[8] The equilibrium solution of (1.2) is the Maxwellian distribution (when $F = 0$), $f_r^o = \nu_r \left(\dfrac{m_r}{2\pi kT}\right)^{3/2} \exp\left(- \dfrac{m_r c^2}{2kT}\right)$. If some components are charged, we must also have: $q = 0$ (neutral fluid) and $E + uxB = 0$.

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(3.1) $f_r^{(k)} = f_r^{i_1 i_2 \cdots i_k} = m_r \nu_r \overline{\xi^{i_1} \xi^{i_2} \cdots \xi^{i_k}} = m_r \nu_r \overline{\xi^{(k)}}$

Instead of considering the _direct_ moments $f_r^{(k)}$, we may use _specific_ _moments_

(3.2) $F_r^{(k)} = m_r \nu_r \overline{c^{(k)}}$ (partial),

(3.2)' $F^{(k)} = \sum_r F_r^{(k)}$ (total)

or the _individual specific moments_

(3.3) $\overline{F}_r^{(k)} = m_r \nu_r \overline{c_r^{(k)}}$.

Some of the moments have been already shown to have a physical meaning. Comparing with the definitions of Sections 1 and 2, we have the following identifications:

$$f_r^O = F_r^O = \overline{F}_r^O = \rho_r \qquad \text{(a scalar)}$$

$$f_r^i = \rho_r u_r, \qquad F_r^i = \rho_r(u_r - u), \qquad \overline{F}_r^i = 0$$

$$F^{ij} = P$$

$$\tfrac{1}{2} F^{ii} = e$$

$$\tfrac{1}{2} F^{ijj} = Q.$$

In the following we shall consider specifically the _partial stress_ _tensor_ P_r, _partial internal energy_ e_r, and _partial heat flow vector_ Q_r defined by

$$P_r = F_r^{ij} ; \qquad p_r = \tfrac{1}{3} F_r^{ii}$$

$$e_r = \tfrac{1}{2} F_r^{ii}$$

$$Q_r = \tfrac{1}{2} F_r^{ijj}$$

These partial moments form the contribution of the component r
to the corresponding observable total moments P, e, and Q. By
analogy with the definition of mixture temperature we define the
component temperature T_r by

$$(D5) \qquad \frac{1}{2} k T_r = \frac{e_r}{3 \nu_r} ,$$

from which follow the relations:

$$(3.4) \qquad p_r = k \nu_r T_r$$

$$(3.5) \qquad \nu T = \sum_r \nu_r T_r .$$

The notation used here differs from that used in MH-I, where
the same symbols are used for individual quantities. The latter
which we now denote by the same letter with a bar, in accordance
with the definition (3.3), are related to the partial moments,
namely:

$$(3.6) \quad \left\{ \begin{array}{l} P_r = \overline{P}_r + \rho_r v_r v_r \\[2mm] e_r = \overline{e}_r + \frac{1}{2} \rho_r v_r^2 \\[2mm] Q_r = \overline{Q}_r + \overline{e}_r v_r + \overline{P}_r \cdot v_r + \frac{1}{2} \rho_r v_r^2 v_r \end{array} \right.$$

where

$$v_r = u_r - u = \text{relative, or diffusion velocity.}$$

The equations that the moments satisfy are obtained from the
equation of transfer. For direct moment equations we get

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$$(3.7) \qquad \frac{\partial}{\partial t} f_r^{(k)} + \frac{\partial}{\partial x_{k+1}} f^{k,k+1} - \sum_{\ell=1}^{k} (F^{i_\ell} + \gamma_r E^{i_\ell}) f_r^{(k/i_\ell)}$$

$$- \sum_{\ell=1}^{k} e_{i_\ell i_{k+1} i_{k+2}} B^{i_{k+2}} f_r^{(k/i_\ell, k+1)} = \sum_s J_{rs}(m_r \xi^{(k)}).$$

It is obvious from their structure that no finite set of moment equations forms a determined set.

While the use of direct moments leads to relativly simpler looking equations, the equations involving individual or partial moments can be interpreted more easily in fluid-dynamical language. We shall make no use here of individual moments. Below we list the 13 moment equations (per component) having for leading term the time derivatives of ρ_r, u_r, P_r and Q_r. The following new symbols are used:

$$(D6) \quad \begin{cases} D = \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial x} \cdot u, \\[2ex] J_{rs}^{(k)} = J_{rs}(m_r c^{(k)}) \\[2ex] F_r = (-\dfrac{\partial}{\partial x} \cdot P + (q - \gamma_r \rho)E + [J - \gamma_r \rho u] \times B) \end{cases}$$

For any second rank tensor a, $\{a\}$ denotes a tensor formed from a by symmetrizing and subtracting its divergence; component-wise,

$$(D7) \qquad \{a^{ij}\} = \frac{1}{2}(a^{ij} + a^{ji}) - \frac{1}{3} a^{kk} \delta_{ij} . \quad [9]$$

The 13n moment equations are then:

---

[9] It is customary to use p^{ij} for $\{P\}$. Then, $P^{ij} = p^{ij} + p\delta_{ij}$. To avoid the superscripts, we use the brace notation.

$$(3.7) \qquad \frac{\partial}{\partial t} f_r^{(k)} + \frac{\partial}{\partial x_{k+1}} f^{k,k+1} - \sum_{\ell=1}^{k} (F^{i_\ell} + \gamma_r E^{i_\ell}) f_r^{(k/i_\ell)}$$

$$- \sum_{\ell=1}^{k} e_{i_\ell i_{k+1} i_{k+2}} B^{i_{k+2}} f_r^{(k/i_\ell, k+1)} = \sum_s J_{rs}(m_r \xi^{(k)}).$$

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$$(3.8) \begin{cases} (C_r) \quad D\rho_r + \frac{\partial}{\partial x} \cdot \rho_r v_r = 0 \\[2mm] (M_r) \quad D\rho_r v_r + \frac{\partial}{\partial x} \cdot P_r + \rho_r v_r \cdot \frac{\partial u}{\partial x} + \frac{\rho_r}{\rho} F_r = \sum_s J^i_{rs} + q_r v_r xB \\[2mm] (E_r) \quad De_r + \frac{\partial}{\partial x} \cdot Q_r + P_r : \frac{\partial u}{\partial x} + \frac{\rho_r}{\rho} F_r \cdot v_r = \frac{1}{2} \sum_s J^{ii}_{rs} \\[2mm] (P_r) \quad D\left\{ P_r \right\} + \left\{ \frac{\partial}{\partial x} F^{(3)}_r \right\} + 2 \left\{ P_r \cdot \frac{\partial u}{\partial x} \right\} - 2\gamma_r \left\{ P_r xB \right\} \\[2mm] \qquad\qquad\qquad\qquad +2 \frac{\rho_r}{\rho} \left\{ F_r v_r \right\} = \left\{ \sum_s J^{ij}_{rs} \right\} \\[3mm] (Q_r) \quad DQ_r + \frac{1}{2} \frac{\partial}{\partial x} \cdot F^{ijkk}_r + Q_r \cdot \frac{\partial u}{\partial x} + F^{(3)}_r : \frac{\partial u}{\partial x} - \gamma_r (Q_r xB) \\[2mm] \qquad\qquad + \frac{1}{\rho}(\frac{3}{2} p_r I + P_r) \cdot F_r = \frac{1}{2} \sum_s J^{ijj}_{rs} \end{cases}$$

The total moment equations are obtained by summing the above over r, except that now we would get an identity by summing the momentum equation. This is because the total momentum equation (2.2) has been already used to simplify the last right hand side term in the equation of transfer, (2.1). We get:

$$(3.9) \begin{cases} (C) \quad D\rho = 0 \\[2mm] (M), (E), \text{ see e.g. } (2:2) \\[2mm] (P) \quad D\left\{ P \right\} + \left\{ \frac{\partial}{\partial x} \cdot F^{(3)} \right\} + 2\left\{ P \cdot \frac{\partial u}{\partial x} \right\} - \sum_r 2\gamma_r \left\{ P_r xB \right\} \\[2mm] \qquad\qquad - 2\left\{ (E + uxB) J_u \right\} = \left\{ \sum_{rs} J^{ij}_{rs} \right\} \\[3mm] (Q) \quad DQ + \frac{1}{2} \frac{\partial}{\partial x} \cdot F^{ijkk} + Q \cdot \frac{\partial u}{\partial x} + F^{(3)} : \frac{\partial u}{\partial x} - \sum_r \gamma_r (Q_r xB) \\[2mm] \qquad\qquad + \frac{1}{\rho}(\frac{3}{2} pI + P) \cdot (- \frac{\partial}{\partial x} \cdot P + qE + JxB) \\[2mm] \qquad\qquad - (E + uxB) \cdot \sum_r \gamma_r (\frac{3}{2} p_r I + P_r) = \frac{1}{2} \sum_{rs} J^{ijj}_{rs} \end{cases}$$

For sake of completeness we also list the 13n moment equation using individual moments:

$$(3.10)\begin{cases} (\overline{M}_r) & \rho_r(\frac{\partial}{\partial t} + u_r \cdot \frac{\partial}{\partial x})u_r + \frac{\partial}{\partial x} \cdot \overline{P}_r = \rho_r F + q_r(E+u_r \times B) + \sum_s \overline{J}^i_{rs} \\[2mm] (\overline{E}_r) & \overline{D}\, \overline{e}_r + \frac{\partial}{\partial x} \cdot \overline{Q}_r + \overline{P}_r : \frac{\partial u_r}{\partial x} = \frac{1}{2}\sum_s \overline{J}^{ii}_{rs} \\[2mm] (\overline{P}_r) & \overline{D}\{\overline{P}_r\} + \{\frac{\partial}{\partial x} \cdot \overline{F}^{(3)}_r\} + 2\{\overline{P}_r \cdot \frac{\partial u_r}{\partial x}\} - 2\gamma_r\{\overline{P}_r \times B\} = \{\sum_s \overline{J}^{ij}_{rs}\} \\[2mm] (\overline{Q}_r) & \overline{D}\, \overline{Q}_r + \frac{1}{2}\frac{\partial}{\partial x} \cdot \overline{F}^{ijkk}_r + \overline{Q}_r \cdot \frac{\partial u_r}{\partial x} + \overline{F}^{(3)}_r : \frac{\partial u_r}{\partial x} - \gamma_r(\overline{Q}_r \times B) \\[2mm] & \quad -(\frac{3}{2}\overline{p}_r I + \overline{P}_r)\cdot(F + \gamma_r E) = \frac{1}{2}\sum_s \overline{J}^{ijk}_{rs} \ . \end{cases}$$

Here,

$$(D8)\begin{cases} \overline{D} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \cdot u_r, \\[3mm] \overline{J}^{(k)}_{rs} = J_{rs}(m_r c^{(k)}_r). \quad [10] \end{cases}$$

4. The 13n moment approximation

The theory of normal solutions leads to the expression of the $(3n+5)$ subsidiary variables v_r, $\{P\}$ and Q^{11} in terms of the primary thermodynamic variables ρ_r, u, and e, thus closing the system (2.2). Here we shall consider as primary all the 13n variables of system

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[10]While any group of equations (3.7), (3.8) and (3.10) may be obtained from any other, it is easiest to proceed in each case directly from the equation of transfer by using consecutively ξ, c and c_r for v.

[11]Since $\sum \rho_r v_r = 0$, there are only $3(n-1)$ independent variables v_r; $\{P\}$ has 5 independent components and Q - three.

(3.8) which are differentiated with respect to time, and shall close the system by approximating in terms of these the remaining terms appearing in (3.8), namely: $F_r^{(3)}$, $F_r^{(4)}$ and the collisional terms, $\sum_s J_{rs}^i$, $\frac{1}{2} \sum_s J_{rs}^{ii}$, $\left\{ \sum_s J_{rs}^{ij} \right\}$, $\frac{1}{2} \sum_s J_{rs}^{ijj}$.[12] These terms are evaluated by choosing an approximation to f_r depending on the primary variables as parameters.

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[12]This is a special case of a broader scheme of "description on various levels." In this scheme one decides a priori which variables (specifically, moments) are relevant for the problem on hand. The full knowledge of the number density function f_r gives the finest description (in the sense of Boltzmann Equation) yielding at once all the moments, and in practice is never desired. The description by means of the $(n+4)$ thermodynamic variables seems to be satisfactory for the equilibrium case, and as noticed at tne conclusion of Section 2, may not be sufficiently detailed in certain non-uniform states of the mixture. In between·we have an unlimited number of possibilities. Our choice is motivated partly by the structure of equations (2.2), which suggests that u_r, P and Q, and with these also the partial stresses and partial heat flows be accepted as primary variables, and partly because these variables have a physical meaning independent of the thermodynamic variables.

For the approximation to f_r we take

$$(4.1) \qquad f_r \sim \widetilde{f}_r = \left(\frac{m_r}{2\pi k T_r} \right)^{3/2} v_r \left[1 + \frac{1}{5} \sqrt{\frac{\rho_r}{p_r^3}} \, (c^2 - 5) c \cdot Q_r \right.$$

$$\left. + \frac{1}{2} \sqrt{\frac{\rho_r}{p_r}} \, (7 - c^2) c \cdot v_r + \frac{1}{2 p_r} cc : \{ P_r \} \right] \exp(- \tfrac{1}{2} c^2),^{[13]}$$

where

$$C = \sqrt{\frac{m_r}{k T_r}} \, c \ .$$

Using (4.1), we now find

---

[13]This choice is motivated by a number of considerations some of which are simplicity considerations. We require that:

   a) $\widetilde{f}_r$ reduces to the equilibrium solution f_r^o (see note 8) when v_r, $\{ P_r \}$ and Q_r vanish

   b) $\widetilde{f}_r$ yields the same value as f_r for the primary moments

   c) $\widetilde{f}_r$ be of the form $\exp(-av^2)$ x polynomial of lowest possible degree in ξ.

We chose here an approximation which depends on the mixture since it involves explicitly the mass velocity u. Another equally plausible choice would be one which satisfies the above conditions and does not take cognizance of the mixture. It would be obtained from (4.1) by replacing all partial moments by individual moments, and c by c_r. The latter would lead, however, to complications in the evaluation of collisional integrals, and is therefore ruled out in favor of the former.

The approximation used is part of a scheme of successive approximations in which f_r is expanded using generalized Hermite polynomials, see [4], with weight $\exp(- \tfrac{1}{2} c^2)$, and the Hermite coefficients from a certain rank on are set equal to zero.

$$(4.2) \quad \begin{cases} F_r^{(3)} \sim \frac{2}{5} \text{ Cyclic sum of } (Q_r I) \\[2mm] F_r^{ijkk} \sim \frac{p_r}{\rho_r} (5 p_r I + 7\{P_r\}). \end{cases} \quad [14]$$

From these we get for the terms appearing in (3.8, P_r, Q_r):

$$(4.3) \quad \begin{cases} \left\{\frac{\partial}{\partial x} \cdot F_r^{(3)}\right\} \sim \frac{4}{5}\left\{\frac{\partial Q_r}{\partial x}\right\} \\[3mm] F_r^{(3)} : \frac{\partial u}{\partial x} \sim \frac{2}{5} (Q_r \frac{\partial}{\partial x} \cdot u + Q_r \cdot \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \cdot Q_r) \\[3mm] \frac{1}{2} \frac{\partial}{\partial x} \cdot F_r^{ijkk} \sim \frac{5}{2} \frac{\partial}{\partial x} \frac{p_r^2}{\rho_r} + \frac{7}{2} \frac{\partial}{\partial x} \cdot (\frac{p_r}{\rho_r} \{P_r\}) \end{cases}$$

To close the system within the present approximation, we still need to evaluate the collisional terms. Formulae for these are listed in Sections 6, 7 and 8.

5. Material constants

We now complete the definition of the collisional term J_{rs}, equation (1.3). In connection with this term one considers a binary

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[14] The approximate expression for $\overline{F}_r^{(3)}$ and $\overline{F}_r^{(4)}$ in terms of the individual moments are complicated and ultimately would lead to the closed system (3.10) that is more complicated than the closed system (3.8). Thus on one hand we are lead to the choice (4.1) for f_r by certain (among others) simplicity considerations, and on the other hand, as a consequence of the preceding, we rule out the description in terms of individual moments in favor of the equivalent description in terms of partial moments.

encounter of a molecule (r, ξ) with a molecule (s, ξ_1) at an impact
parameter b. We assume that the molecules act on each other with
a force derived from a potential $U_{rs}(d)$, where d is their distance.
Let w be the relative velocity

(D9) $w = \xi_1 - \xi$,

and

(D10) $m_{rs} = \dfrac{m_r m_s}{m_r + m_s}$

the reduced mass. The encounter will produce a deflection $\mathcal{X}$
in the relative velocity

(5.1) $\mathcal{X} = \pi - 2\theta$,

where,

(5.2) $\theta(|w|, b) = \displaystyle\int_0^{x_{max}} \left(1 - x^2 - \dfrac{2}{w^2 m_{rs}} U(\dfrac{b}{x})\right)^{-1/2} dx.$

(x_{max} is the value of x for which the integrand becomes infinite;
it is related to the distance of closest approach.)

One shows that the terminal velocities of the two molecules are

(5.3) $\begin{cases} \xi' = \xi + 2\,\dfrac{m_{rs}}{m_r}\,(w \cdot a)a \\[2mm] \xi_1' = \xi_1 - 2\,\dfrac{m_{rs}}{m_r}\,(w \cdot a)a, \end{cases}$

and the differential collisional cross-section $d\Omega$ is

(5.4) $d\Omega = |w| b\, db\, d\varepsilon = V_{rs} d\theta d\varepsilon$, $V_{rs} = |w| b\, \dfrac{\partial b}{\partial \theta} = V_{rs}(|w|, \theta)$.

Here a is a unit vector in the direction of the apse line to the
trajectory in a system moving with one of the molecules, and makes

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the angle $-\theta$ with the direction of w; ε is the angle that the plane of the trajectory makes with a fixed direction in the impact plane. The limits of integration are (for a monotone potential): $0 \leq b < \infty$, $0 \leq \varepsilon < 2\pi$, or alternately, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \varepsilon < 2\pi$.

The collisional terms in (3.8) are evaluated in terms of the relevant moments by using $\tilde{f}_r$ in place of f_r in (1.3). The results of integration involve certain material constants (depending only on the nature and the temperature of interacting components), A_{rs}^k and B_{rs}^k, $k = 1,2,\ldots$, defined as follows:

(5.5)
$$\begin{cases} A_{rs}^k = \dfrac{1}{1 \cdot 3 \ldots (2k+1)} \sqrt{\dfrac{2}{\pi}} \int_0^\infty x^{2(k+1)} A_{rs}\left(x \sqrt{\dfrac{kT_r}{m_r} + \dfrac{kT_s}{m_s}}\right) e^{-\frac{x^2}{2}} dx \\[4mm] B_{rs}^k = \dfrac{1}{1 \cdot 3 \ldots (2k+1)} \sqrt{\dfrac{2}{\pi}} \int_0^\infty x^{2(k+1)} B_{rs}\left(x \sqrt{\dfrac{kT_r}{m_r} + \dfrac{kT_s}{m_s}}\right) e^{-\frac{x^2}{2}} dx, \end{cases}$$

where

(5.6)
$$\begin{cases} A_{rs}(|w|) = \dfrac{4\pi}{m_r + m_s} \int_0^{\pi/2} V_{rs}(|w|,\theta)\cos^2\theta\, d\theta \quad [15] \\[4mm] B_{rs}(|w|) = \dfrac{4\pi}{m_r + m_s} \int_0^{\pi/2} V_{rs}(|w|,\theta)\cos^2\theta \sin^2\theta\, d\theta. \end{cases}$$

---

[15] In various sources the deflection χ is used instead of the apse line angle θ. In terms of χ, $d\Omega = |w| b \frac{db}{d\chi} d\chi d\varepsilon = -\frac{1}{2} V_{rs}(|w|, \frac{\pi - \chi}{2}) d\chi d\varepsilon$,

$$\int_0^{\pi/2} V_{rs} \cos^2\theta\, d\theta = \frac{1}{4} \int_0^\pi V_{rs}(1 - \cos\chi) d\chi. \qquad \int_0^{\pi/2} V_{rs}\cos^2\theta \sin^2\theta\, d\theta$$

$$= \frac{1}{8} \int_0^\pi V_{rs}(1 - \cos^2\chi) d\chi.$$

The dimension of A_{rs}^k and B_{rs}^k is $L^3M^{-1}T^{-1}$.

For various molecular models the material constants can be evaluated explicitly. We quote the results:

5-1. <u>Molecules repelling according to an inverse μ-th power law</u>

$$U_{rs} = \frac{\varkappa_{rs}}{\mu-1} x^{-\mu+1},$$

$$V_{rs} = |w|^\eta \left(\frac{\varkappa_{rs}}{m_{rs}}\right)^{(1-\eta)/2} \beta \frac{d\beta}{d\theta}, \qquad \eta = \frac{\mu-5}{\mu-1},$$

$$\text{where} \quad \theta(\beta) = \int_0^{x_{max}} (1 - x^2 - \frac{2}{\mu-1} (\frac{x}{\beta})^{\mu-1})^{-1/2} dx;$$

$$A_{rs}^k = \frac{(5+\eta)(7+\eta) \ldots (2k+1+\eta)}{5\cdot 7 \ldots (2k+1)} A_{rs}',$$

$$B_{rs}^k = \frac{(5+\eta)(7+\eta) \ldots (2k+1+\eta)}{5\cdot 7 \ldots (2k+1)} B_{rs}',$$

where

$$A_{rs}' = A_{rs}'(\eta) = \frac{8\sqrt{2\pi}}{3} \Gamma(\frac{5+\eta}{2}) \mathcal{A}_1 (m_r+m_s)^{-1} \left(\frac{\varkappa_{rs}}{m_{rs}}\right)^{(1-\eta)/2} \left(\frac{kT_r}{m_r} + \frac{kT_s}{m_s}\right)^{\eta/2},$$

$$B_{rs}'(\eta) = \frac{4\sqrt{2\pi}}{3} \Gamma(\frac{5+\eta}{2}) \mathcal{A}_2 (m_r+m_s)^{-1} \left(\frac{\varkappa_{rs}}{m_{rs}}\right)^{(1-\eta)/2} \left(\frac{kT_r}{m_r} + \frac{kT_s}{m_s}\right)^{\eta/2},$$

and

$$\mathcal{A}_1(\eta) = 2 \int_0^{\pi/2} \beta \frac{d\beta}{d\theta} \cos^2\theta d\theta,$$

$$\mathcal{A}_2(\eta) = 4 \int_0^{\pi/2} \beta \frac{d\beta}{d\theta} \cos^2\theta \sin^2\theta d\theta.$$

$\mathcal{A}_1$ and $\mathcal{A}_2$ are tabulated for selected values of $\eta \geq 5$ in [1], p. 172.

For $\mu = 5$, $\eta = 0$ and V_{rs} does not depend on $|w|$. In this case, the collisional integrals can be evaluated in terms of moments without the explicit knowledge of f_r. Notice that in this case $A_{rs}^k = A_{rs}'$, $B_{rs}^k = B_{rs}'$ for all k.

For $\mu \leq 2$, A_{rs}' and B_{rs}' both diverge.

Similar results can be established for attracting molecules.

5-2. Molecules acting like perfectly elastic spheres

This is a limiting case of the preceding with $\mu = \infty$, $\eta = 1$, and the relation between A_{rs}^k and A_{rs}' etc., is the same.

$V_{rs} = d_{rs}^2 |w| \sin \theta \cos \theta$, $\quad d_{rs} = \frac{1}{2}(d_r + d_s)$, where the d's are
the molecular diameters

$$A_{rs}' = 3B_{rs}' = \frac{8\sqrt{2\pi}}{3}(m_r + m_s)^{-1} d_{rs}^2 \sqrt{\frac{kT_r}{m_r} + \frac{kT_s}{m_s}}$$

5-3. Charged molecules

Charged molecules interact according to Coulomb's law ($\mu = 2$) for which the integrals $\mathcal{A}_1$ and $\mathcal{A}_2$ diverge. This indicates that the assumption of binary collisions (or binary forces) for charged particles is too crude to yield significant results. In view of footnote 4 we assume that long range Coulomb force effects are taken care of by addition of electromagnetic terms to the drift terms of the Boltzmann Equation, and that binary collisions occur only when the interacting molecules are within certain shielding distance d_o.

For d_o we take the Debye length,

$$d_o = \left(4\pi \sum_r \frac{\varepsilon_r q_r}{kT_r} \right)^{-1/2} ,$$

see [6], which depends on the temperatures of the fluid. The interaction potential is thus assumed to be

$$U_{rs} = \begin{cases} \dfrac{\varepsilon_r \varepsilon_s}{d} & \text{for } d \leq d_o \\[2ex] 0 & \text{for } d > d_o. \end{cases} \text{[16]}$$

We find:

$$V_{rs} = \begin{cases} |w|^{-3} \dfrac{\varepsilon_r \varepsilon_s}{m_{rs}}^2 \sin\theta \cos^{-3}\theta & \text{for } 0 \leq \theta \leq \theta_o \leq \text{arc tan} \dfrac{m_{rs} w^2 d_o}{|\varepsilon_r \varepsilon_s|} \\[2ex] 0 & \text{for } \theta_o \leq \theta \leq \pi/2 \end{cases}$$

$$A_{rs}^k = \frac{\sqrt{2\pi}}{m_r + m_s} \left(\frac{\varepsilon_r \varepsilon_s}{m_{rs}} \right)^2 \left(\frac{kT_r}{m_r} + \frac{kT_s}{m_s} \right)^{-3/2} a_{rs}^k$$

$$B_{rs}^k = \frac{\sqrt{2\pi}}{m_r + m_s} \left(\frac{\varepsilon_r \varepsilon_s}{m_{rs}} \right)^2 \left(\frac{kT_r}{m_r} + \frac{kT_s}{m_s} \right)^{-3/2} b_{rs}^k$$

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[16] The Yukawa potential

$$\frac{\varepsilon_r \varepsilon_s}{d} \exp(-\frac{d}{d_o})$$

has been suggested instead of the Coulomb with cut-off (R. Liboff, report to appear). The results are not very different in a high temperature plasma. In [8] it is suggested that a short range potential cut-off be used to take account of quantum effects. This seems to introduce only insignificant corrections, in particular for relatively large Debye lengths, i.e. when temperatures are high and densities low.

where, for small

$$z = \frac{|\epsilon_r \epsilon_s|}{2m_{rs}d_o(\dfrac{kT_r}{m_r} + \dfrac{kT_s}{m_s})}$$

$$a^k_{rs} = \frac{2^{k+1}(k-1)!}{1 \cdot 3 \ldots (2k+1)} \left(\log \frac{1}{z} + k - 1 - \gamma\right) \quad (\gamma = .577 \ldots, \text{ the Euler constant}).$$

$$b^k_{rs} = \frac{2^{k+1}(k-1)!}{1 \cdot 3 \ldots (2k+1)} \left(\log \frac{1}{z} + \frac{1}{2}k - 1 - \gamma\right)$$

These results seem to be either little known or inaccessible and are therefore derived in the Appendix. As shown there, the quantities a^k and b^k can be expressed in closed form in terms of tabulated functions, and are tabulated for various values of z. Estimates of material constants will be found in Section 8.

6. Formulae for collisional terms

The formulae for J^i_{rs}, etc., listed below were derived on the basis of calculation in [7][17]. In this list we replace the letters r and s by 1 and 2. We also drop the subscripts from the material constants. The collisional terms with general indices may be obtained from those presented here by adjusting the subscripts. We use the following symbols:

_____

[17]The results presented in [7] are not expressed explicitly in terms of moments. Needless to say, their derivation is long and tedious though not basically difficult. In any involved calculations of this sort mistakes are likely. Our results were rechecked directly several times by the author and by Dr. Carl Oberman who found a few misprints in [7]. Several consistency checks were also made. These are based on the following:

1. The formulae must reduce to the exact results for the case of Maxwellian molecules. The latter can be evaluated (more ->

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$$(D11) \begin{cases} m_{10} = \dfrac{m_1}{m_1+m_2} \ , \quad m_{20} \ \dfrac{m_2}{m_1+m_2} \\[2mm] v_{12} = v_1 - v_2 \\[2mm] \tau = \left(\dfrac{kT_1}{m_1} + \dfrac{kT_2}{m_2} \right)^{-1} \\[2mm] \sigma = \dfrac{k(T_1-T_2)}{m_1+m_2} \tau \\[2mm] \widetilde{Q}_r = Q_r - \dfrac{5}{2} P_r v_r \end{cases}$$

$$(6.1) \quad J_{12}^i = -A^1 \rho_1 \rho_2 v_{12} - (A^2-A^1) \tau (\rho_2[\widetilde{Q}_1 - \{P_1\} \cdot v_2] - \rho_1[\widetilde{Q}_2 - \{P_2\} \cdot v_2])$$

$$- \frac{7}{5}(A^3 - 2A^2 + A^1) \tau^2 (\{P_2\} \cdot \widetilde{Q}_1 - \{P_1\} \cdot \widetilde{Q}_2)$$

$$(6.2) \quad \frac{1}{2} J_{12}^{ii} = -3\frac{\sigma}{\tau} A^1 \rho_1 \rho_2 + (5\sigma (A^2-A^1) + (m_{10}-m_{20})A^1) \rho_1 \rho_2 \ v_1 \cdot v_2$$

$$+ 2(A^2-A^1) \tau^2 (\rho_2 v_2 \cdot \widetilde{Q}_1 - \rho_1 v_1 \cdot \widetilde{Q}_2)$$

$$+\left(\sigma(7A^3-12A^2+5A^1) + (m_{10}-m_{20})(A^2-A^1) \right) \tau (\rho_2 v_2 \cdot \widetilde{Q}_1 +$$

$$+\rho_1 v_1 \cdot \widetilde{Q}_2)$$

$$+(\frac{7}{2}\sigma (A^3-2A^2+A^1)+(m_{10}-m_{20})(A^2-A^1) \tau \ \{P_1\} : \{P_2\}$$

$$+ \frac{21}{5}\left(3\sigma (A^4-3A^3+3A^2-A^1)+(m_{10}-m_{20})(A^3-2A^2+A^1) \right) \tau^2 \widetilde{Q}_1 \cdot \widetilde{Q}_2$$

---

without employing an expansion of the distribution function.

2. The formulae must reduce to those obtained in [3] for the single gas when either the subscripts are set equal, or the mixture is made of identical components with $v_r=k_r v$, $P_r=k_r P$, $Q_r=k_r Q$, $\sum k_r=1$, and $v_r=0$, $T_r=T$. The latter implies that a single gas is looked on as a mixture of its parts.

$$(6.3) \quad \left\{J_{12}^{ij}\right\} = -(2\sigma(A^2-A^1) + 2m_{10}A^1 + 3m_{20}B^2)\rho_2\left\{P_1\right\}$$

$$-(2\sigma(A^2-A^1) - 2m_{20}A^1 + 3m_{20}B^2)\rho_1\left\{P_2\right\}$$

$$+2(2\sigma(A^2-A^1) + (m_{10}-m_{20})A^1 + 3m_{20}B^2)\rho_1\rho_2\left\{v_1v_2\right\}$$

$$+ \frac{8}{5}(A^2-A^1)\tau^2\left\{P_2v_2\tilde{Q}_1 - P_1v_1\tilde{Q}_2\right\}$$

$$+ \frac{2}{5}\Big(2\sigma(7A^3-12A^2+5A^1) + 5(m_{10}-m_{20})(A^2-A^1)$$

$$+21m_{20}(B^3-B^2)\Big)\tau\left\{P_2v_2\tilde{Q}_1 + P_1v_1\tilde{Q}_2\right\}$$

$$-2(2\sigma(A^3-2A^2+A^1) + (m_{10}-m_{20})(A^2-A^1)+3m_{20}(B^3-B^2))\tau\left\{P_1\cdot P_2\right\}$$

$$+ \frac{126}{25}\Big(2\sigma(A^4-3A^3+3A^2-A^1)+(m_{10}-m_{20})(A^3-2A^2+A^1)$$

$$+ 3m_{20}(B^4-2B^3+B^2)\Big)\tau^2\left\{\tilde{Q}_1\tilde{Q}_2\right\}$$

$$(6.4) \quad \frac{1}{2}J_{12}^{ijj} = -5\frac{\sigma}{\tau}A^1\rho_1\rho_2[(m_{10}-\sigma)v_1 + (m_{20}+\sigma)v_2]$$

$$- \frac{5}{2\tau}\Big((m_{10}-\sigma)(m_{20}+\sigma)A^1+(3\sigma^2+m_{20}^2)A^2+4\sigma\,m_{20}B^2\Big)\rho_1\rho_2v_{12}$$

$$-\Big(2\sigma(A^2-A^1) - 2m_{20}A^1+3m_{20}B^2\Big)\rho_1v_1\cdot\left\{P_2\right\}$$

$$-\Big(2\sigma(A^2-A^1)+2m_{10}A^1 + 3m_{20}B^2\Big)\rho_2v_2\cdot\left\{P_1\right\}$$

$$-\Big((3m_{10}^2+ m_{20}^2)A^1 + \frac{1}{2}(3\sigma^2+m_{20}^2)(7A^3-5A^2-2A^1)$$

$$+ \frac{5}{2}(m_{10}-\sigma)(m_{20}+\frac{27}{5}\sigma)(A^2-A^1) + 14\sigma\,m_{20}(B^3-B^2)$$

$$+ 4m_{10}m_{20}B^2\Big)\rho_2(\tilde{Q}_1- \left\{P_1\right\}\cdot v_2)$$

$$+ \Big(4m_{20}^2A^1 + \frac{1}{2}(3\sigma^2 + m_{20}^2)(7A^3 - 5A^2 - 2A^1)$$

$$+ \frac{5}{2}(m_{20}+\sigma)(m_{10}-\frac{27}{5}\sigma)(A^2-A^1) + 14\sigma\,m_{20}(B^3-B^2)$$

$$- 4m_{20}^2B^2\Big)\rho_1(\tilde{Q}_2 - \left\{P_2\right\}\cdot v_1).$$

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$$- \Bigg((m_{20}+ \tfrac{9}{5}m_{10}- \tfrac{4}{5}\sigma)(A^2-A^1) + \tfrac{7}{10}(m_{10}-\sigma)(m_{20}+\sigma)(5A^3-22A^2+17A^1)$$

$$+ \tfrac{7}{10}(3\sigma^2+m_{20}^2)(9A^4-14A^3+5A^2)+ \tfrac{14}{5}\sigma m_{20}(9B^4-14B^3+5B^2)$$

$$+ \tfrac{2}{5}\sigma (28A^3-33A^2+5A^1)- \tfrac{7}{5}\sigma (m_{20}+\sigma)(17A^3- 22A^2+ 5A^1)$$

$$- \tfrac{7}{5}m_{20}(m_{20}- 7m_{10}+ 8\sigma)(B^3- B^2) \Bigg) \tau \ \{P_2\}\cdot \tilde{Q}_1$$

$$+ \Bigg((m_{10}+ \tfrac{9}{5}m_{20} + \tfrac{4}{5}\sigma)(A^2-A^1)+\tfrac{7}{10}(m_{10}-\sigma)(m_{20}+\sigma)(5A^3-22A^2+17A^1)$$

$$+ \tfrac{7}{10}(3\sigma^2+m_{20}^2)(9A^4-14A^3+5A^2)+ \tfrac{14}{5}\sigma m_{20}(9B^4-14B^3+5B^2)$$

$$- \tfrac{2}{5}\sigma (28A^3-33A^2+5A^1)+ \tfrac{7}{5}\sigma (m_{10}-\sigma)(17A^3-22A^2+5A^1)$$

$$+ \tfrac{7}{5} m_{20}(m_{10}- 7m_{20}- 8\sigma)(B^3-B^2) \Bigg) \tau \ \{P_1\}\cdot \tilde{Q}_2$$

7. Linear collisional terms

In many situations the products of the moments u_r, P_r and Q_r can be neglected when compared to themselves. A finer analysis[18] shows that the temperature differences $(T-T_r)$ are comparable to two term products of these moments. Making this assumption we set

$$(7.1) \quad \begin{cases} \tau \sim \dfrac{m_{12}}{kT} \\[2mm] \sigma \sim 0 \, , \end{cases}$$

except in J_{12}^{ii} where no linear terms appear.

We now get the following expressions for the collisional terms:

---

[18]This is discussed in [7] and will be further elaborated on in a future note.

$$(7.2) \qquad J^i_{12} \sim - \left(A^1 - \frac{5}{2} m_{20}(A^2 - A^1)\right) \rho_1 \rho_2 v_1 + \left(A^1 - \frac{5}{2} m_{10}(A^2 - A^1)\right) \rho_1 \rho_2 v_2$$

$$-(A^2 - A^1) \frac{m_{12}}{kT} (\rho_2 Q_1 - \rho_1 Q_2)$$

$$(7.3) \qquad \frac{1}{2} J^{ii}_r = - \frac{3A^1 \rho_1 \rho_2}{m_1 + m_2} k(T_1 - T_2) + \frac{1}{2}(m_{10} - m_{20}) \left((7A^1 - 5A^2)\right.$$

$$+ \frac{105}{2} m_{10} m_{20} (A^3 - 2A^2 + A^1) \Big) \rho_1 \rho_2 v_1 \cdot v_2$$

$$- (m_{10} - m_{20})(A^2 - A^1) \frac{m_{12}}{kT} \{P_1\} : \{P_2\}$$

$$- \frac{m_{10} m_{12}}{kT} \left((\frac{m_2}{m_1} - 3)(A^2 - A^1) + \frac{21}{2}(m_{10} - m_{20})(A^3 - 2A^2 + A^1)\right) \rho_2 v_2 \cdot Q_1$$

$$- \frac{m_{20} m_{12}}{kT} \left((3 - \frac{m_1}{m_2})(A^2 - A^1) + \frac{21}{2}(m_{10} - m_{20})(A^3 - 2A^2 + A^1)\right) \rho_1 v_1 \cdot Q_2$$

$$+ \frac{21}{5}(m_{10} - m_{20}) \left(\frac{m_{12}}{kT}\right)^2 Q_1 \cdot Q_2$$

$$(7.4) \qquad \{J^{ij}_{12}\} = -(2m_{10}A^1 + 3m_{20}B^2)\rho_2 \{P_1\} - m_{20}(3B^2 - 2A^1)\rho_1 \{P_2\}$$

$$(7.5) \qquad J^{ijj}_{12} = -\left(3m_{10}^2 A^1 + \frac{5}{2} m_{10} m_{20}(A^2 - A^1) + \frac{1}{2} m_{20}^2 (7A^3 - 5A^2) + 4 m_{10} m_{20} B^2\right) \rho_2 Q_1$$

$$+ m_{20}\left(\frac{5}{2} m_{10}(A^2 - A^1) + \frac{1}{2} m_{20}(7A^3 - 5A^2 + 6A^1) - 4 m_{20} B^2\right) \rho_1 Q_2$$

$$+ \frac{5kT}{2(m_1 + m_2)} \left(2 m_{10} A^1 + \frac{1}{2} m_{20}(3A^2 - 7A^1) + \frac{7}{2} \frac{m_{20}^2}{m_{10}}(A^3 - A^2)\right.$$

$$+ 4 m_{20} B^2\Big) \rho_1 \rho_2 v_1$$

$$- \frac{5kT}{2(m_1 + m_2)} \left(\frac{1}{2} m_{10}(5A^2 - 7A^1) + \frac{1}{2} m_{20}(7A^3 - 7A^2 + 4A^1) - \frac{m_{20}^2}{m_{10}} A^2\right.$$

$$- 4 m_{20} B^2\Big) \rho_1 \rho_2 v_2$$

8. Collisional terms for a plasma

In ionized gases an approximation which exploits the fact that $\frac{m}{M} \ll 1$, is in order; here m is the electron mass, and M the ion mass. We shall assume that the two temperatures as well as the number densities, are comparable. Using the subscript (-) to refer to electrons, and (+) to ions, our statement is that $\frac{T_+}{T_-}$ and $\frac{\nu_+}{\nu_-}$ are neither very small nor very large. On the other hand,

$$(8.1) \quad \begin{cases} \dfrac{kT_-}{m} \gg \dfrac{kT_+}{M} \ , \\[2ex] \rho_- = m\nu_- \ll M\nu_+ = \rho_+ \ . \end{cases}$$

With these assumptions

$$(8.2) \quad \begin{cases} \tau_{--} = \dfrac{m}{2kT_-} \\[2ex] \tau_{-+} = \tau_{+-} = \dfrac{m}{kT_-} \\[2ex] \tau_{++} = \dfrac{M}{2kT_+} \\[2ex] \sigma_{-+} = \dfrac{k(T_- - T_+)}{M} = -\sigma_{+-} \end{cases}$$

(Of course, $\sigma_{--} = \sigma_{++} = 0$).

We first discuss the material constants. Denote by ε the electron charge, by Z the ionization number. Thus $\varepsilon_- = -\varepsilon$, and $\varepsilon_+ = Z\varepsilon$. Then

$$(8.3) \quad \begin{cases} z_{--} = \dfrac{\varepsilon^2}{2kT_- d_o} \\[2ex] z_{-+} = z_{+-} = \dfrac{\varepsilon^2 Z}{2kT_- d_o} \\[2ex] z_{++} = \dfrac{\varepsilon^2 Z^2}{2kT_+ d_o} \end{cases}$$

Thus the z's are essentially of the same order of magnitude and their logarithms will differ little[19], and the same thus follows for the constants a^k and b^k. (See Section 5.3).

Let

$$(8.4) \qquad \chi = \sqrt{\pi} \; \varepsilon^4 \, (kmT_-)^{-3/2}.$$

From formulae of Section 5.3 we get

$$(8.5) \qquad \begin{cases} A^k_{--} = \chi \, a^k_{--} \\[2ex] A^k_{-+} = A^k_{+-} = z^2 \, \dfrac{m}{M} \, \sqrt{2} \, \chi \, a^k_{-+} \\[2ex] A^k_{++} = z^4 \left(\dfrac{mT_-}{MT_+} \right)^{3/2} \chi \, a^k_{++} \, , \end{cases}$$

with similar formulae for the B^k.

We now use the collisional terms of Section 6 assuming that products of u_r, P_r and Q_r may be neglected[20], but not assuming that $T_- = T_+$. One obtains

$$(8.6) \qquad J^i_{-+} = -z^2 \, \frac{m}{M} \, \sqrt{2} \, \chi \left(\left(a^1 - \frac{5}{2}(a^2 - a^1) \right) \rho_- \rho_+ v_- - a^1 \rho_- \rho_+ v_+ \right.$$

$$\left. + \frac{m}{kT_-} (a^2 - a^1)(\rho_+ + Q_- - \rho_- Q_+) \right)$$

$$(J^i_{--} = J^i_{++} = 0, \quad J^i_{+-} = -J^i_{-+}).$$

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[19] The Debye length can be estimated by the formula $d_o = \dfrac{kT}{2\varepsilon\sqrt{\pi p}}$, which assumes that the component temperatures are equal and the ions are singly ionized. The estimate for z is then $z_{--} \sim \dfrac{\varepsilon^3 \sqrt{\pi p}}{(kT)^2}$. For $T = 10^6 \, {}^\circ K$ and $p = 100$ atm, $z_{--} \sim 10^{-4}$, $\text{Log} \frac{1}{z} \sim 9$.

[20] This assumption may turn out to be inappropriate in certain cases.

(8.7) $\quad \frac{1}{2} J^{ii}_{-+} = -z^2 \frac{m}{M} \sqrt{2} \chi \left(3a^1 \frac{k(T_- - T_+)}{M} \rho_-\rho_+ + \left(a^1 - \frac{5}{2}(a^2 - a^1) \right) \rho_-\rho_+ v_- \cdot v_+ \right.$

$$- \frac{m}{kT_-}(a^2 - a^1)\{P_-\} : \{P_+\} + \frac{m}{kT_-}(a^2 - a^1)\rho_+ v_+ \cdot Q_-$$

$$\left. - \frac{3m}{kT_-}(7a^3 - 16a^2 + 9a^1)\rho_- v_- \cdot Q_+ + \frac{21}{5}\left(\frac{m}{kT_-}\right)^2 (a^3 - 2a^2 + a^1)Q_1 \cdot Q_2 \right)$$

$(J^{ii}_{--} = J^{ii}_{++} = 0, \ J^{ii}_{+-} = -J^{ii}_{-+}).$

(8.8) $\quad \left\{ J^{ij}_{--} + J^{ij}_{-+} \right\} = -\chi m \left((3v_- b^2_{--} + 3\sqrt{2} \, z^2 v_+ \, b^2)\{P_-\} \right.$

$$\left. + \sqrt{2} \, z^2 \frac{m}{M} v_-(3b^2 - 2a^1)\{P_+\} \right)$$

(8.9) $\quad \left\{ J^{ij}_{+-} + J^{ij}_{++} \right\} = -\sqrt{2} \, z^2 \frac{m}{M} \chi m \left(v_+(2 \frac{T_+}{T_-}(a^2 - a^1) + 3b^2 - 2a^2)\{P_-\} \right.$

$$\left. + (2v_- a^1 + 3z^2\sqrt{\frac{M}{2m}} \left(\frac{T_-}{T_+}\right)^{3/2} v_+ b^2_{++})\{P_+\} \right)$$

(8.10) $\quad \frac{1}{2}(J^{ijj}_{--} + J^{ijj}_{-+}) = -\chi m \left((2v_- b^2_{--} + \frac{\sqrt{2}}{2} z^2 v_+ (7a^3 - 5a^2))Q_- \right.$

$$- \sqrt{2} \, z^2 \frac{m}{M} v_- \left(3a^1 + \frac{1}{2}(7a^3 - 5a^2) - 4b^2 \right) Q_+$$

$$- kT_- v_- \left(5v_- b^2_{--} + \frac{35\sqrt{2}}{4} z^2 v_+ (a^3 - a^2) \right) v_-$$

$$\left. - \frac{5\sqrt{2}}{2} z^2 kT_- v_- v_+ a^2 v_+ \right)$$

(8.11) $\quad \frac{1}{2}(J^{ijj}_{+-} + J^{ijj}_{++}) = -\sqrt{2} \, z^2 \frac{m}{M} \chi m \left(-\frac{5T_+}{2T_-} v_+ (a^2 - a^1)Q_- \right.$

$$+ \left(3v_- a^1 + 2z^2\sqrt{\frac{M}{2m}} \left(\frac{T_-}{T_+}\right)^{3/2} v_+ b^2_{++} \right) Q_+$$

$$\left. - \frac{5}{2} kT_+ v_- v_+ \left(a^1 - \frac{5}{2}(a^2 - a^1) \right) v_- \right.$$

$$- 5kT_-v_+(v_- a^1 + Z^2 \sqrt{\frac{MT_-}{2mT_+}} \; v_+ b^2_{++}) \; v_+ \Bigg).$$

In the above a^k and b^k (without subscript) stand for a^k_{-+} and b^k_{-+} .

Appendix

Evaluation of Material Constants for Charged Components

Using the cross-section V_{rs} of Section 5-3, we find for the integrals (5.6)

(A1)
$$\begin{cases} A_{rs} = \dfrac{2\pi}{m_r + m_s} \left(\dfrac{\varepsilon_r \varepsilon_s}{m_{rs}} \right)^2 |w|^{-3} \log(1 + \zeta w^4) \\[3mm] B_{rs} = \dfrac{2\pi}{m_r + m_s} \left(\dfrac{\varepsilon_r \varepsilon_s}{m_{rs}} \right)^2 |w|^{-3} \left(\log(1 + \zeta w^4) - \dfrac{\zeta w^4}{1 + \zeta w^4} \right), \end{cases}$$

where

$$\zeta = \left(\frac{m_{rs} d_o}{\varepsilon_r \varepsilon_s} \right)^2.$$

Substituting (A1) in (5.5) we get, using a new variable of integration, $y = \dfrac{x^2}{2}$,

(A2)
$$a_{rs}^k = \frac{2^k}{1 \cdot 3 \ldots (2k+1)} \int_o^\infty y^{k-1} e^{-y} \log\left(1 + \left(\frac{y}{z}\right)^2\right) dy,$$

(A3)
$$b_{rs}^k = \frac{k+2}{2} a^k - \frac{2k+3}{4} a^{k+1}.$$

The second of these is obtained from the original formula by integration by parts.

In evaluating the integral in (A2) we observe that since both z and y are positive,

$$\frac{1}{2} \log(z^2 + y^2) = \operatorname{Re}\big(\log(z + yi)\big).$$

Hence

$$(A4) \quad \int_0^\infty y^{k-1} e^{-y} \log\left(1+\left(\frac{y}{z}\right)^2\right) dy = -2(k-1)!\log z$$

$$+ 2 \operatorname{Re} \int_0^\infty y^{k-1} e^{-y} \log(z+yi) dy$$

$$= -2(k-1)!\left(\log z + \operatorname{Re} \sum_{p=0}^{k-1} \frac{y^p}{p!} e^{-y} \log(z+yi)\Big|_0^\infty\right.$$

$$\left. - \operatorname{Re} i \int_0^\infty \frac{\sum_0^{k-1} \frac{y^p}{p!} e^{-y}}{z+yi} dy\right)$$

$$= 2(k-1)! \operatorname{Re} \sum_{p=0}^{k-1} \frac{i}{p!} \int_0^\infty \frac{y^p e^{-y}}{z+yi} dy,$$

after integrating by parts.

Now, using the identity

$$\frac{y^p}{z+yi} \equiv \frac{1}{i}\left(\frac{y^p - (iz)^p}{y - iz} + \frac{(iz)^p}{y-iz}\right) = \frac{(iz)^p}{z+iy} - i \sum_{g=0}^{p-1} y^q (iz)^{p-1-q},$$

we get

$$\int_0^\infty \frac{y^p e^{-y}}{z+yi} dy = (iz)^p \int_0^\infty \frac{e^{-y} dy}{z+iy} - i \sum_{q=0}^{p-1} q!(iz)^{p-1-q}.$$

Thus the integration is reduced to the evaluation of $\int_0^\infty \frac{e^{-y}}{z+iy} dy$. This integral can be considered as a complex integral

$$\int_\uparrow \frac{e^{i(\zeta-z)}}{\zeta} \frac{d\zeta}{i},$$

the path of integration running along the positive part of the line parallel to the imaginary axis with abscissa z. The path of integration can be now deformed into a path running along the real axis, from z to ∞. Since now the variable of integration is again real,

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we can return to real notation and get

$$\int_0^\infty \frac{e^{-y}}{z+iy}\, dy = -i \int_z^\infty \frac{e^{i(\sigma-z)}}{\sigma}\, d\sigma = -i \int_0^\infty \frac{e^{i\sigma}}{\sigma+z}\, d\sigma$$

$$= -i(C(z) + iS(z)),$$

where

$$(A5) \quad \begin{cases} C(z) = \int_0^\infty \frac{\cos\sigma}{\sigma+z}\, d\sigma, \\[2em] S(z) = \int_0^\infty \frac{\sin\sigma}{\sigma+z}\, d\sigma. \end{cases}$$

Substituting in (A4) we now get after some manipulations with the series

$$(A6) \quad \int_0^\infty y^{k-1}\, e^{-y} \log\left(1+\left(\frac{y}{z}\right)^2\right) dy = 2(k-1)! \ \mathrm{Re}\ (C+iS) \sum_{p=0}^{k-1} \frac{(iz)^p}{p!}$$

$$+ \sum_{q=0}^{k-2} (iz)^q \sum_{p=q}^{r-2} \frac{(p-q)!}{p!} \quad.$$

Both $C(z)$ and $S(z)$ can be expanded in the form

Power series + log z Power series.

We get:

$$(A7) \quad \begin{cases} S = \frac{\pi}{2} + z \log z + O(z) \\[1em] C = -\log z - \gamma + O(z \log z). \end{cases}$$

Keeping in (A6) only the terms up to order $O(1)$, and substituting in (A2,3), we now get the formulae quoted in the text, Section 5.3.

Since for $x > 0$, $\log(1+x) \le x$,

$$\int_0^\infty y^{k-1} e^{-y} \log\left(1+\left(\tfrac{y}{z}\right)^2\right) dy \leq z^{-2} \int_0^\infty y^{k+1} e^{-y} dy = \frac{(k+1)!}{z^2}.$$

Hence as $z \to \infty$, both a^k and b^k tend to zero as $\dfrac{1}{z^2}$.

In terms of $C(z)$ and $S(z)$, we get, using (A6) in (A2,3), the exact formulae for the a^k, b^k:

$$(A8) \quad \begin{cases} a^1 = \dfrac{4}{1\cdot 3}\, C \\[2mm] a^2 = \dfrac{8}{1\cdot 3\cdot 5}(1 + C - zS) \\[2mm] a^3 = \dfrac{16\cdot 2}{1\cdot 3\cdot 5\cdot 7}\left(2 + \left(1 - \tfrac{z^2}{2}\right)C - zS\right) \\[2mm] a^4 = \dfrac{32\cdot 6}{1\cdot 3\cdot 5\cdot 7\cdot 9}\left(\left(3 - \tfrac{z^2}{2}\right) + \left(1 - \tfrac{z^2}{2}\right)C - \left(z - \tfrac{z^3}{6}\right)S\right) \end{cases}$$

$$(A9) \quad \begin{cases} b^1 = \dfrac{4}{1\cdot 3}\left(-\tfrac{1}{2} + C + \tfrac{1}{2}zS\right) \\[2mm] b^2 = \dfrac{8}{1\cdot 3\cdot 5}\left(\left(1 + \tfrac{z^2}{2}\right)C - zS\right) \\[2mm] b^3 = \dfrac{16\cdot 2}{1\cdot 3\cdot 5\cdot 7}\left(\left(\tfrac{1}{2} + \tfrac{3z^2}{4}\right) + \left(1 - \tfrac{z^2}{2}\right)C - \left(z + \tfrac{z^3}{4}\right)S\right) \\[2mm] b^4 = \dfrac{32\cdot 6}{1\cdot 3\cdot 5\cdot 7\cdot 9}\left(\left(1 - \tfrac{z^2}{6}\right) + \left(1 - \tfrac{z^2}{2} - \tfrac{z^4}{12}\right)C - \left(z - \tfrac{z^2}{6}\right)S\right) \end{cases}$$

The functions $C(z)$ and $S(z)$ can be expressed in terms of the sine integral (Si) and cosine integral (Ci) tabulated in [9], p. 6 ff. We get

$$(A10) \quad \begin{cases} C(z) = -\cos z\, Ci(z) + \sin z\left(\tfrac{\pi}{2} - Si(z)\right) \\[2mm] S(z) = \sin z\, Ci(z) + \cos z\left(\tfrac{\pi}{2} - Si(z)\right) \end{cases}$$

The values of the a's and the b's are tabulated below together with their approximate values given by formulae of Section 5.3.

z	Exact	Approximate	Exact	Approximate
	a^1		b^1	
.01	5.39	5.38	4.73	4.71
.1	2.49	2.29	1.91	1.63
1	.458	- .769	.205	-1.44
	a^2		b^2	
.01	2.68	2.68	2.14	2.14
.1	1.46	1.45	.930	.920
1	.384	.225	- .057	- .308
	a^3		b^3	
.01	1.84	1.84	1.38	1.38
.1	1.14	1.14	.682	.677
1	.472	.434	.197	- .023
	a^4		b^4	
.01	1.43	1.43	1.020	1.020
.1	.958	.958	.554	.553
1	.437	.492	.093	.086

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